Time Dependent Tempered Generalized Functions and Itô's Formula

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Abstract

The paper introduces a novel Itô's formula for time dependent tempered generalized functions. As an application, we study the heat equation when initial conditions are allowed to be a generalized tempered function. A new proof of the Ustunel- Itô's formula for tempered distributions is also provided.

Key words: Generalized functions, Itô's formula, Stochastic calculus via regularization, Hermite expansions.

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1 Introduction

The study of Stochastic Partial Differential Equations (SPDE) in algebras of generalized functions could be traced back to the early 1990s, see [1], [2], [11] and [17]. The interest for singular stochastic processes, such as the white noise process, and solve differential equations driven by these type of processes are the main reasons for treating of SPDE in the framework of the algebras of generalized functions (see by example [1] and [4]).
The central theme of the present paper is to develop stochastic calculus via regularization in the setting of algebras of generalized functions. The main technique is the construction of an algebra of tempered generalized functions via the regularization scheme induced by expansions in Hermite functions. This approach allows us to obtain an Itô's formula for elements in the algebra. In particular, we deduce the Ustunel- Itô's formula (see [19]) for tempered distributions (see also [9] and [14]). Sections 2 and 3 of the present paper present the relevant precise definitions and details.
As an application of our results, we also show the existence and uniqueness of the solution to the heat equation with the initial condition being a tempered generalized function; this makes crucial use of the obtained Itô's formula for tempered generalized functions. See Sections 4 for details.

2 Generalized functions

2.1 Tempered distributions

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space on $\mathbb{R}^d$ i.e. the space of rapidly decreasing smooth real valued functions on $\mathbb{R}^d$.

We make use of the multi-index notation; a multi-index is a sequence $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$ where $\mathbb{N}_0$ is the set of nonnegative integers. The sum $|\alpha| = \sum_{j=1}^n \alpha_j$ is called the order of α. For every multi-index α we write

$$x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d}$$

and

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$$

where $\partial_j = \frac{\partial}{\partial x_j}$.

The Schwartz topology on $\mathcal{S}(\mathbb{R}^d)$ is given by the family of seminorms

$$\|f\|_{\alpha,\beta} = (\int_{\mathbb{R}^d} |x^\alpha \partial^\beta f(x)|^2 dx)^{\frac{1}{2}}$$

where $\alpha, \beta \in \mathbb{N}_0^d$.

The Schwartz space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions is the dual space of $\mathcal{S}(\mathbb{R}^d)$.

The *Hermite polynomials* $H_n(x)$ are defined by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2}} \tag{1}$$

and the *Hermite functions* $h_n(x)$ are defined by

$$h_n(x) = (\sqrt{2\pi} n!)^{-\frac{1}{2}} e^{-\frac{1}{4}x^2} H_n(x) \tag{2}$$

for $n \in \mathbb{N}_0$.

The α-th. Hermite function on $\mathbb{R}^d$ is given by

$$h_\alpha(x_1, ..., x_d) = h_{\alpha_1}(x_1) \cdots h_{\alpha_d}(x_d)$$

where $\alpha = (\alpha_1, ..., \alpha_d) \in \mathbb{N}_0^d$.

The Hermite functions are in the Schwartz space on $\mathbb{R}^d$ and the set $\{h_\alpha : \alpha \in \mathbb{N}_0^d\}$ is an orthonormal basis for $L^2(\mathbb{R}^d)$.

We consider the directed family of norms $\{|\cdot|_n : n \in \mathbb{N}_0\}$ on $\mathcal{S}(\mathbb{R}^d)$, given by

$$|\varphi|_n^2 := \sum_{\beta \in \mathbb{N}_0^d} (2|\beta| + d)^{2n} (\int_{\mathbb{R}^d} \varphi(x) h_\beta(x) dx)^2.$$

We observe that the families of seminorms $\{|\cdot|_n : n \in \mathbb{N}_0\}$ and $\{\|\cdot\|_{\alpha,\beta} : \alpha, \beta \in \mathbb{N}_0^d\}$ on $\mathcal{S}(\mathbb{R}^d)$ are equivalent.

Let $x \in \mathbb{R}^d$, and denote by τ_x the translation operator defined on functions by the formula $\tau_x\varphi(y) = \varphi(y - x)$. It follows immediately that $\tau_x(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$ and that τ_{-x} is the inverse of τ_x, thus, we can consider τ_x acting on the tempered distributions by

$$\tau_x T(\varphi) = T(\tau_{-x}\varphi).$$

Lemma 2.1 *Let $n \in \mathbb{N}_0$, there exists a polynomial P_n with nonnegative coefficients such that for all $x \in \mathbb{R}^d$,*

$$|\tau_x\varphi|_n \leq P_n(|x|)|\varphi|_n \tag{3}$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Proof: Using Proposition 3.3 from [15], for each $n \in \mathbb{N}_0$ there exist constants $C_1(n)$ and $C_2(n)$ such that

$$|\varphi|_n \leq C_1(n) \sum_{|\alpha|,|\beta| \leq 2n} \|\varphi\|_{\alpha,\beta} \leq C_2(n)|\varphi|_n$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

Thus

$$
\begin{aligned}
|\tau_x\varphi|_n &\leq C_1(n) \sum_{|\alpha|,|\beta|\leq 2n} (\int_{\mathbb{R}^d} y^{2\alpha}(\partial^\beta \tau_x\varphi)^2(y)dy)^{\frac{1}{2}} \\
&= C_1(n) \sum_{|\alpha|,|\beta|\leq 2n} (\int_{\mathbb{R}^d} (y+x)^{2\alpha}(\partial^\beta \varphi)^2(y)dy)^{\frac{1}{2}} \\
&\leq P_n(|x|)|\varphi|_n,
\end{aligned}
$$

where P_n is a polynomial of degree lower or equal to $2n$.

Multiplication on $\mathcal{S}(\mathbb{R}^d)$ has the following property: for all $n \in \mathbb{N}_0$ there exists $r, s \in \mathbb{N}_0$ and $C_n \in \mathbb{R}$ such that

$$|\varphi\psi|_n \leq C_n|\varphi|_r|\psi|_s \tag{4}$$

for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ (see for instance [13]). We shall make often use of this property.

The Hermite representation theorem for $\mathcal{S}(\mathbb{R}^d)$ ($\mathcal{S}'(\mathbb{R}^d)$) states a topological isomorphism beetwen $\mathcal{S}(\mathbb{R}^d)$ ($\mathcal{S}'(\mathbb{R}^d)$) and the space of sequences $\mathbf{s}_d$ ($\mathbf{s}'_d$).

Let $\mathbf{s}_d$ be the space of sequences

$$\mathbf{s}_d = \{(a_\beta) \in \ell^2(\mathbb{N}^d)) : \sum_{\beta\in\mathbb{N}_0^d} (2|\beta|+d)^{2n} \mid a_\beta \mid^2 < \infty, \text{ for all } \ n \in \mathbb{N}_0\}.$$

The space $\mathbf{s}_d$ is a locally convex space with the family of norms

$$|(a_\beta)|_n = (\sum_{\beta\in\mathbb{N}_0^d} (2|\beta|+d)^{2n} \mid a_\beta \mid^2)^{\frac{1}{2}},$$

where $n \in \mathbb{N}_0$.

The topological dual space to $\mathbf{s}_d$, denoted by $\mathbf{s}'_d$, is given by

$$\mathbf{s}'_d = \{(b_\beta) : \text{for some } \ (C,m) \in \mathbb{R}\times\mathbb{N}_0^d, \mid b_\beta \mid \leq C(2|\beta|+d)^m \text{ for all } \beta\},$$

and the natural pairing of elements from $\mathbf{s}_d$ and $\mathbf{s}_d'$, denoted by $\langle \cdot, \cdot \rangle$, is given by

$$\langle (b_\beta), (a_\beta) \rangle = \sum_{\beta \in \mathbb{N}_0^d} b_\beta a_\beta,$$

for $(b_\beta) \in \mathbf{s}_d'$ and $(a_\beta) \in \mathbf{s}_d$.

It is clear that $\mathbf{s}_d'$ is an algebra with the pointwise operations:

$$\begin{aligned} (b_\beta) + (b_\beta') &= (b_\beta + b_\beta') \\ (b_\beta) \cdot (b_\beta') &= (b_\beta b_\beta'), \end{aligned}$$

and $\mathbf{s}_d$ is an ideal of $\mathbf{s}_d'$.

Theorem 2.1 (N-representation theorem for $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$) **a**) *Let* $\mathbf{h} : \mathcal{S}(\mathbb{R}^d) \to \mathbf{s}_d$ *be the application*

$$\mathbf{h}(\varphi) = (\int \varphi(x) h_\beta(x) dx).$$

Then $\mathbf{h}$ *is a topological isomorphism. Moreover,*

$$|\mathbf{h}(\varphi)|_n = |\varphi|_n$$

for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$.

b) *Let* $\mathbf{H} : \mathcal{S}'(\mathbb{R}^d) \to \mathbf{s}_d'$ *be the application* $\mathbf{H}(T) = (T(h_\beta))$. *Then* $\mathbf{H}$ *is a topological isomorphism. Moreover, if* $T \in \mathcal{S}'(\mathbb{R}^d)$ *we have that*

$$T = \sum_{\beta \in \mathbb{N}_0^d} T(h_\beta) h_\beta$$

in the weak sense and for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$T(\varphi) = \langle \mathbf{H}(T), \mathbf{h}(\varphi) \rangle.$$

Proof: See for instance [16] pp. 143 or [18] pp. 260.

The sequences $\mathbf{h}(\varphi)$ and $\mathbf{H}(T)$ will be referred to as the *Hermite coefficients* of the tempered function φ and the tempered distribution T respectively.

Corollary 2.2 *For every $T \in \mathcal{S}'(\mathbb{R}^d)$ there exists $n \in \mathbb{N}_0$, such that*

$$\|T\|_{-n}^2 := \sum_{\beta \in \mathbb{N}_0^d} (2|\beta| + d)^{-2n} T(h_\beta)^2 < \infty.$$

Proof: By Theorem 2.1, $(T(h_\beta)) \in \mathbf{s}'_d$. Thus, there exists $(C, l) \in \mathbb{R} \times \mathbb{N}_0$ such that $|T(h_\beta)| \leq C(2|\beta| + d)^l$ for all $\beta \in \mathbb{N}_0^d$. Now, taking $n = l + 1$ the Corollary follows.

2.2 Tempered generalized functions

The aim of this subsection is to give an extension to the multidimensional case of the theory of tempered generalized functions introduced by the authors in [3]. Let $\mathcal{S}_T^1(\mathbb{R}^d)$ be the set of functions $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$ such that for each $t \in [0, T]$, $f(t, \cdot) \in \mathcal{S}(\mathbb{R}^d)$ and for each $x \in \mathbb{R}^d$, $f(\cdot, x) \in C^1([0, T])$. It is clear that $\mathcal{S}_T^1(\mathbb{R}^d)^{\mathbb{N}_0^d}$ has the structure of an associative, commutative differential algebra with the natural operations:

$$\begin{aligned}
(f_\beta) + (g_\beta) &:= (f_\beta + g_\beta) \\
a(f_\beta) &:= (a f_\beta) \\
(f_\beta) \cdot (g_\beta) &:= (f_\beta g_\beta) \\
\partial^\alpha (f_\beta) &:= (\partial_x^\alpha f_\beta) \text{ for each } \alpha \in \mathbb{N}_0^d.
\end{aligned}$$

In order to define the 1-time dependent tempered generalized functions, we consider $\mathcal{H}'_{T,1,d}$ the subalgebra of $\mathcal{S}_T^1(\mathbb{R}^d)^{\mathbb{N}_0^d}$ given by

$$\{(f_\beta) \in \mathcal{S}_T^1(\mathbb{R}^d)^{\mathbb{N}_0^d} : \text{for each } n \in \mathbb{N}_0, (\sup_{t \in [0,T]} |f_\beta(t, \cdot)|_n), (\sup_{t \in [0,T]} |\frac{\partial f_\beta}{\partial t}(t, \cdot)|_n) \in \mathbf{s}'_d\}$$

and $\mathcal{H}_{T,1,d}$ its differential ideal given by

$$\{(f_\beta) \in \mathcal{S}_T^1(\mathbb{R}^d)^{\mathbb{N}_0^d} : \text{for each } n \in \mathbb{N}_0, (\sup_{t \in [0,T]} |f_\beta(t, \cdot)|_n), (\sup_{t \in [0,T]} |\frac{\partial f_\beta}{\partial t}(t, \cdot)|_n) \in \mathbf{s}_d\}.$$

The 1-time dependent tempered algebra on $\mathbb{R}^d$ is defined by

$$\mathcal{H}_T^1(\mathbb{R}^d) := \mathcal{H}'_{T,1,d}/\mathcal{H}_{T,1,d}$$

The elements of $\mathcal{H}_T^1(\mathbb{R}^d)$ are called 1-time dependend tempered generalized functions. Let $(f_\beta) \in \mathcal{H}'_{T,1,d}$ we shall use $[f_\beta]$ to denote the equivalent class $(f_\beta) + \mathcal{H}_{T,1,d}$.

Remark 2.1 *In order to introduce the* 0*-dependent tempered generalized functions, we consider* $\mathcal{S}_T^0(\mathbb{R}^d)$ *the of the set of functions* $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ *such that for each* $t \in [0,T]$, $f(t,\cdot) \in \mathcal{S}(\mathbb{R}^d)$ *and for each* $x \in \mathbb{R}^d$, $f(\cdot,x) \in C([0,T])$*. Let* $\mathcal{H}'_{T,0,d}$ *be the subalgebra given by*

$$\{(f_\beta) \in \mathcal{S}_T^0(\mathbb{R}^d)^{\mathbb{N}_0^d} : \textit{for each } n \in \mathbb{N}_0,\ (\sup_{t\in[0,T]} |f_\beta(t,\cdot)|_n) \in \mathbf{s}'_d\}$$

and $\mathcal{H}_{T,0,d}$ *its differential ideal given by*

$$\{(f_\beta) \in \mathcal{S}_T(\mathbb{R}^d)^{\mathbb{N}_0^d} : \textit{for each } n \in \mathbb{N}_0,\ (\sup_{t\in[0,T]} |f_\beta(t,\cdot)|_n) \in \mathbf{s}_d\}.$$

The 0*-time dependent tempered algebra on* $\mathbb{R}^d$ *is defined by*

$$\mathcal{H}_T^0(\mathbb{R}^d) := \mathcal{H}'_{T,0,d}/\mathcal{H}_{T,0,d}$$

Remark 2.2 *In a similar way we can define the tempered algebra*

$$\mathcal{H}(\mathbb{R}^d) := \mathcal{H}'_d/\mathcal{H}_d$$

where

$$\mathcal{H}'_d := \{(f_\beta) \in \mathcal{S}(\mathbb{R}^d)^{\mathbb{N}_0^d} : \textit{for each } n \in \mathbb{N}_0,\ (|f_\beta|_n) \in \mathbf{s}'_d\}$$

and

$$\mathcal{H}_d := \{(f_\beta) \in \mathcal{S}(\mathbb{R}^d)^{\mathbb{N}_0^d} : \textit{for each } n \in \mathbb{N}_0,\ (|f_\beta|_n) \in \mathbf{s}_d\}.$$

The elements of $\mathcal{H}(\mathbb{R}^d)$ *are called tempered generalized functions.*

Proposition 2.3 *1. $\mathcal{H}(\mathbb{R}^d)$ is a subalgebra of $\mathcal{H}_T^0(\mathbb{R}^d)$ ($\mathcal{H}_T^1(\mathbb{R}^d)$).*

2. Let $[f_\beta] \in \mathcal{H}_T^1(\mathbb{R}^d)$. Then

$$\frac{\partial}{\partial t}[f_\beta(t,\cdot)] := [\frac{\partial f_\beta}{\partial t}(t,\cdot)] \in \mathcal{H}^0(\mathbb{R}^d)$$

for every $t \in [0,T]$.

3. Let $h \in C^1([0,T])$ and $[f_\beta] \in \mathcal{H}(\mathbb{R}^d)$. Then $h[f_\beta] := [hf_\beta] \in \mathcal{H}_T^1(\mathbb{R}^d)$.

Proof: The proofs are straigthforward from the definitions.

We observe that there exists a natural linear embedding $\iota : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{H}(\mathbb{R}^d)$, given by

$$\iota(T) = [T_\beta],$$

where $T_\beta = \sum_{\gamma \leq \beta} T(h_\gamma)h_\gamma$. Moreover, we have that

a) For all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\iota(\varphi) = [\varphi]$,

b) For all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, $\iota(\varphi\psi) = \iota(\varphi) \cdot \iota(\psi)$,

c) For all $\alpha \in \mathbb{N}_0^d$, $\iota \circ \partial^\alpha = \partial^\alpha \circ \iota$.

The translation operator $\tau_x : \mathcal{H}_T^0(\mathbb{R}^d) \to \mathcal{H}_T^0(\mathbb{R}^d)$ ($x \in \mathbb{R}^d$) is defined by

$$\tau_x[f_\beta] := [\tau_x f_\beta].$$

It follows from Lemma 1.1 that τ_x is well defined. Analogously, $\tau_x : \mathcal{H}_T^1(\mathbb{R}^d) \to \mathcal{H}_T^1(\mathbb{R}^d)$ ($x \in \mathbb{R}^d$) is well defined.

In the algebra $\mathcal{H}(\mathbb{R}^d)$ we have a weak equality, namely, the association of tempered generalized functions. More precisely, we say that the tempered generalized functions $[f_\beta]$ and $[g_\beta]$ are associated, and denote this association by $[f_\beta] \approx [g_\beta]$, if for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$\lim_{\beta \to \infty} \int_{\mathbb{R}^n} (f_\beta(x) - g_\beta(x))\varphi(x)dx = 0.$$

We observe that $\approx$ is a equivalence relation on $\mathcal{H}(\mathbb{R}^d)$.

Proposition 2.4 *1. Let T be a tempered distribution and $x \in \mathbb{R}^d$. Then $\iota(\tau_x T) \approx \tau_x \iota(T)$.*

2. Let T and S be tempered distributions such that $\iota(T) \approx \iota(S)$. Then $T = S$.

Proof: 1) Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$; applying Theorem 2.1 and elementary properties of the translation, we obtain

$$\begin{aligned}
\lim_{\beta\to\infty} \int_{\mathbb{R}^d} \tau_x T_\beta(y)\varphi(y)dy &= \lim_{\beta\to\infty} \int_{\mathbb{R}^d} T_\beta(y)\tau_{-x}\varphi(y)dy \\
&= T(\tau_{-x}\varphi) \\
&= \tau_x T(\varphi) \\
&= \lim_{\beta\to\infty} \int_{\mathbb{R}^n} (\tau_x T)_\beta(y)\varphi(y)dy.
\end{aligned}$$

2) For all $\alpha \in \mathbb{N}_0^d$,

$$T(h_\alpha) - S(h_\alpha) = \lim_{\beta\to\infty} T_\beta(h_\alpha) - S_\beta(h_\alpha) = 0.$$

Theorem 2.1 implies that $T = S$.

Remark 2.3 *We would like to recall that he algebra $\mathcal{H}(\mathbb{R}^d)$ is identical to (a sequential version of) the space $G_{S(\mathbb{R}^d)}$, it has been studied extensively by G. Garetto and their coauthors (see [6], [7] and [8]). They introduced the space $G_{S(\mathbb{R}^d)}$ via the family of seminorms $\{\|\cdot\|_{\alpha,\beta,\infty} : \alpha, \beta \in \mathbb{N}_0^d\}$. Notice that it does not matter which family of seminorms is being used in this definition, as long as it generates the same locally convex topology (see [6]).*
Our approach differs, we have introduced the algebra $\mathcal{H}(\mathbb{R}^d))$ via the seminorms $\| \ \|_m$ since we are thinking in approximations of distributions (induced by Hilbert spaces) in terms of orthogonal series in contrast with the classic theory where the approximation is done by convolution. See [4] for an application of this idea to stochastic distributions.

Remark 2.4 *We would like to mention that other general properties of $\mathcal{H}(\mathbb{R}^d)$ can be studied in this setting. For example the concepts of point value, integral and Fourier transform for elements in $\mathcal{H}(\mathbb{R}^d)$ can be defined, see [3] for the one-dimensional case.*

3 Itô's formula for tempered generalized functions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0,T]\}, \mathbb{P})$ be a filtered probability space, which satisfies the usual hypotheses. For a recent account of stochastic calculus we refer the reader to the book of Ph. Protter [12].

Definition 3.1 *Let X be a $\mathbb{R}^d$ valued continuous jointly measurable process, V a continuous finite variation process and $[f_\beta] \in \mathcal{H}_T^0(\mathbb{R}^d)$. Define the integral of $\tau_X[f_\beta]$ in relation to V, from 0 to t, and denoted by $\int_0^t \tau_{X_s}[f_\beta]dV_s$, by:*

$$[\int_0^t \tau_{X_s} f_\beta(s,\cdot)dV_s],$$

where the integral is given in the sense of Bochner-Stieltjes.

For each $\omega \in \Omega$ and $t \in [0,T]$, we have that $[\int_0^t \tau_{X_s} f_\beta(s,\cdot)dV_s(\omega)]$ is well defined as an element of $\mathcal{H}_T^0(\mathbb{R}^d)$. In fact, since $\tau_{X_s(\omega)} f_\beta(s,\cdot) \in \mathcal{S}_T^0(\mathbb{R}^d)$ and making use of definitions and the Lemma 2.1 we see that

$$\begin{aligned} |\int_0^t \tau_{X_s} f_\beta(s,\cdot)dV_s(\omega)|_n &\leq \int_0^t |\tau_{X_s(\omega)} f_\beta(s,\cdot)|_n d|V|_s(\omega) \\ &\leq (\int_0^t P_n(|X_s(\omega)|)d|V|_s(\omega)) \sup_{s\in[0,T]} |f_\beta(s,\cdot)|_n \quad (5) \end{aligned}$$

where $|V|_t(\omega)$ is the total variation of V in $[0,t]$.

Theorem 3.1 *Let $f = [f_\beta] \in \mathcal{H}^1_T(\mathbb{R}^d)$ and $X = (X^1, ..., X^d)$ be a $\mathbb{R}^d$ valued continuous semimartingale. Then*

$$\begin{aligned} \tau_{X_t} f &= \tau_{X_0} f + \int_0^t \partial_t \tau_{X_s} f(s,\cdot)ds - \int_0^t \nabla \tau_{X_s} f \cdot dX_s \\ &\quad + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}\tau_{X_s} f d\langle X^i, X^j\rangle_s \end{aligned} \tag{6}$$

where $\int_0^t \nabla\tau_{X_s} f \cdot dX_s$ is defined as $[\sum_{i=1}^d \int_0^t \partial_i \tau_{X_s} f_\beta dX^i_s]$.

Proof: Applying the classical Itô's formula to f_β, we have

$$\begin{aligned} \tau_{X_t} f_\beta(t,x) &= \tau_{X_0} f_\beta(0,x) + \int_0^t \partial_t \tau_{X_s} f_\beta(s,x)ds - \sum_{i=1}^d \int_0^t \partial_i \tau_{X_s} f_\beta(s,x) dX^i_s \\ &\quad + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}\tau_{X_s} f_\beta(s,x)\ d\langle X^i, X^j\rangle_s. \end{aligned} \tag{7}$$

Taking equivalent classes in (7), we obtain that $\int_0^t \nabla\tau_{X_s} f \cdot dX_s \in \mathcal{H}^0_T(\mathbb{R}^d)$ and hence (6) holds in $\mathcal{H}^0_T(\mathbb{R}^d)$.

Corollary 3.2 *Let $f = [f_\beta] \in \mathcal{H}(\mathbb{R}^d)$ and $X = (X^1, ..., X^d)$ be a continuous semimartingale. Then*

$$\tau_{X_t} f = \tau_{X_0} f - \int_0^t \nabla \tau_{X_s} f \cdot dX_s + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}\tau_{X_s} f d < X^i, X^j >_s .$$

3.1 Itô's formula for tempered distributions

In order to prove the Üstunel-Itô's formula for tempered distributions (see [19]), we need the following result from the stochastic integration theory in nuclear spaces (see [20]). Let T_t be a $\mathcal{S}'(\mathbb{R}^d)$-valued continuous predictable process and X_t be a continuous semimartingale, then $\int_0^t T_s dX_s$ is the unique $\mathcal{S}'(\mathbb{R}^d)$ valued semimartingale such that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$(\int_0^t T_s dX_s)(\varphi) = \int_0^t T_s(\varphi) dX_s.$$

Lemma 3.1 *Let $T \in \mathcal{S}'(\mathbb{R}^d)$, X be a $\mathbb{R}^d$ valued continuous semimartingale and V be a continuous finite variation process. Then*

$$\int_0^t \tau_{X_s}\iota(T)dV_s \approx \iota(\int_0^t \tau_{X_s}TdV_s).$$

Proof: Let $\iota(T) = [T_\beta]$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since $\lim_{\beta\to\infty} T_\beta = T$ we have

$$\lim_{\beta\to\infty} \tau_{X_s(\omega)}T_\beta(\varphi) = \tau_{X_s(\omega)}T(\varphi)$$

for all s and $\omega \in \Omega$.

Applying the Corollary 2.2 and Lemma 2.1 we have that there exists $q \in \mathbb{N}_0$ such that

$$\begin{aligned} |\tau_{X_s(\omega)}T_\beta(\varphi)| &\leq |T_\beta|_{-q}|\tau_{X_s(\omega)}\varphi|_q \\ &\leq P_q(|X_s(\omega)|)\ |T|_{-q}\ |\varphi|_q. \end{aligned}$$

By the dominate convergence Theorem we obtain

$$\lim_{\beta\to\infty}\int_0^t \tau_{X_s}T_\beta(\varphi)dV_s = \int_0^t \tau_{X_s}T(\varphi)dV_s.$$

We conclude that

$$\begin{aligned} \lim_{\beta\to\infty}\int_0^t \tau_{X_s(\omega)}\iota(T)dV_s(\omega)_\beta(\varphi) &= \lim_{\beta\to\infty}\int_0^t \tau_{X_s(\omega)}T_\beta(\varphi)dV_s(\omega) \\ &= \int_0^t \tau_{X_s(\omega)}T(\varphi)dV_s(\omega) \\ &= \lim_{\beta\to\infty}\iota(\int_0^t \tau_{X_s(\omega)}TdV_s(\omega))_\beta(\varphi), \end{aligned}$$

and the proof is complete.

Proposition 3.3 (Itô's formula for tempered distributions) *Let $T \in \mathcal{S}'(\mathbb{R}^d)$ and $X = (X^1, ..., X^d)$ be a continuous semimartingale. Then*

$$\tau_{X_t}T = \tau_{X_0}T - \sum_{i=1}^d \int_0^t \partial_i\tau_{X_s}TdX_s^i + \frac{1}{2}\sum_{i,j=1}^d \int_0^t \partial_{ij}\tau_{X_s}Td < X^i, X^j >_s .$$

Proof: Applying Itô's formula (6) to $\iota(T)$ and making use of Proposition 2.4 and Lemma 3.1 we deduce that

$$\iota(\tau_{X_t}T-\tau_{X_0}T-\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t\partial_{ij}\tau_{X_s}fd<X^i,X^j>_s)\approx\int_0^t\nabla\tau_{X_s}\iota(T)\cdot dX_s.$$

According to an easy modification of Lemma 3.1 (we make use of the dominated convergence theorem for stochastic integrals) we can to prove that

$$\int_0^t\nabla\tau_{X_s}\iota(T)\cdot dX_s\approx\iota(\sum_{i=1}^{d}\int_0^t\partial_i\tau_{X_s}TdX_s^i). \tag{8}$$

Then, from the Proposition 2.4 and equality (8) we conclude the proof.

4 Heat equation in $\mathcal{H}_T^1(\mathbb{R}^d)$

We introduce next the concept of expected value (or expectation) for certain $\mathcal{H}(\mathbb{R}^d)$-valued random variables. More precisely, let X be a $\mathbb{R}^d$ valued random variable with $\mathbb{E}(|X|^n)<\infty$ for all $n\in\mathbb{N}_0$ and let $f=[f_\beta]\in\mathcal{H}(\mathbb{R}^d)$. The expectation of $\tau_X f$ is $\mathbb{E}(\tau_X f):=[\mathbb{E}(\tau_X f_\beta)]$.

We observe that $\mathbb{E}(\tau_X f)$ is well-defined as an element of $\mathcal{H}(\mathbb{R}^d)$. In fact, by Lemma 2.1, it follows that:

$$|\mathbb{E}(\tau_X f_\beta)|_n \;\leq\; \mathbb{E}(P_n(|X|))|f_\beta|_n,$$

for all $\beta\in\mathbb{N}_0^d$.

The remaining of the present section is concerned with the Cauchy problem for the heat equation,

$$\begin{cases} u_t &= \frac{1}{2}\triangle u \\ u_0 &= f\in\mathcal{H}(\mathbb{R}^d). \end{cases} \tag{9}$$

Definition 4.1 *We say that* $u \in \mathcal{H}_T^1(\mathbb{R}^d)$ *is a generalized solution of the Cauchy problem (9) if* $u_t = \frac{1}{2}\triangle u$ *in* $\mathcal{H}_T^0(\mathbb{R}^d)$ *and* $u_0 = f$ *in* $\mathcal{H}(\mathbb{R}^d)$*.*

Proposition 4.1 *For every* $f \in \mathcal{H}(\mathbb{R}^d)$ *there exist a unique solution to the Cauchy problem (9) in* $\mathcal{H}_T^1(\mathbb{R}^d)$*.*

Proof: Step 1 (Existence) By Itô's formula (6) we have

$$\tau_{B_t} f = f - \int_0^t \nabla \tau_{B_s} f \cdot dB_s + \int_0^t \frac{1}{2}\triangle \tau_{B_s} f ds. \tag{10}$$

We observe that: $\mathbb{E}(\int_0^t \triangle \tau_{B_s} f ds) = \int_0^t \mathbb{E}(\triangle \tau_{B_s} f) ds$; in fact, this is a consequence of inequality (5), that $\mathbb{E}(|B_t|^n) < \infty$ and that $\mathbb{E}(\int_0^t |B_s|^n ds) < \infty$ for all $n \in \mathbb{N}_0$.

Taking expectation in (10) we obtain that

$$\mathbb{E}(\tau_{B_t} f) = f + \int_0^t \frac{1}{2}\triangle \mathbb{E}(\tau_{B_s} f) ds.$$

Thus $\mathbb{E}(\tau_{B_t} f)$ solves the Cauchy problem (9).
Step 2 (Uniqueness) We consider the uniqueness. Suppose that $u = [u_\beta]$ and $v = [v_\beta]$ are two generalized solutions of (9), we denote the difference $u_\beta - v_\beta$ by a_β . By the definition, a_β satisfies

$$\left\{ \begin{array}{lcl} \frac{d}{dt} a_\beta & = & \frac{1}{2}\triangle a_\beta + h_\beta \\ a_\beta(0,\cdot) & = & g_\beta, \end{array} \right. \tag{11}$$

with $h_\beta \in \mathcal{H}_{T,0,d}$ and $g_\beta \in \mathcal{H}_d$. Applying the FeynmanKac formula (see [5]) to a_β we get

$$a_\beta(t,x) \quad = \quad \widetilde{\mathbb{E}}(g_\beta(x + \widetilde{B}_t) + \int_0^t h_\beta(s, x + \widetilde{B}_s)\ ds) \tag{12}$$

where $\widetilde{B}$ is a d-dimensional Brownian motion with $\widetilde{B}_0 = 0$ in an auxiliary probability space.

It follows that $\sup_t \|a_\beta(t,x)\|_n \in \mathbf{s}_d$, for each $n \in \mathbb{N}_0$. From this fact and equation (11) we have that $\sup_t \|\frac{d}{dt} a_\beta(t,x)\|_n \in \mathbf{s}_d$. We conclude that $(a_\beta) \in \mathcal{H}_{T,1,d}$ and thus (9) has an unique solution.

Remark 4.1 *The proof of existence and uniqueness of Proposition 4.1 can be extended easily to the following Cauchy problem,*

$$\begin{cases} u_t &= \frac{1}{2}\triangle u + g, \\ u_0 &= f \in \mathcal{H}(\mathbb{R}^d) \end{cases} \tag{13}$$

where $g \in \mathcal{H}_T^0(\mathbb{R}^d)$.

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